Taylor's Theorem

Let \( f(z) \) be analytic in a region \( G \) containing the point \( z_0 \). Then the representation

\[
f(z) = f(z_0) + f'(z_0)(z-z_0) + \cdots + \frac{f^{(n)}(z_0)(z-z_0)^n}{n!} + \cdots
\]

holds (is convergent) in all disks \( |z-z_0| < r \) contained in \( G \).

**Proof:** Let \( Z \) be an interior point of the closed disk \( 1 - |Z_0| < r \) contained in \( G \), and use the Cauchy integral formula to write

\[
f(z) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(\xi)}{\xi-z} \, d\xi
\]
We have

\[ f(z) = \frac{1}{2\pi i} \int \frac{f(z')}{z-z'} \frac{1}{1 - \frac{z-z_0}{z-z_0}} \, dz' \]

with \( |z-z_0| > |z-z_0| \)

or \( \left| \frac{z-z_0}{z-z_0} \right| < 1 \)

\[ f(z) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z')}{z-z'} \frac{1}{1 - \frac{z-z_0}{z-z_0}} \, dz' \]

\[ = \frac{1}{2\pi i} \int \frac{f(z')}{z-z'} \left[ 1 + \frac{z-z_0}{z-z_0} + \left( \frac{z-z_0}{z-z_0} \right)^2 + \cdots + \left( \frac{z-z_0}{z-z_0} \right)^{n-1} + Q_n \right] \, dz' \]

where \( Q_n = \frac{(z-z_0)^n}{(z-z_0)^n-1} \)

\[ = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z')}{(z-z_0)^n} \, dz' + \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z')}{(z-z_0)^n} \, dz' \]

\[ f(z_0) + R_n \]
and $R_n = (z - z_0)^n \frac{f(z)}{2\pi i} \int \frac{dz}{(z - z)^n}$ is the remainder term of the series.

Need to get a bound on $R_n$, i.e., show $R_n \to 0$ as $n \to \infty$

Let $|z - z_0| = \rho \rightarrow \rho < r = |z - z_0|

Let $M$ be maximum of $|f(z)|$ on $|z - z_0| = r$

Now $|z - z| = |z - z_0 + z_0 - z| > |z - z_0| - |z - z_0| = r - \rho

$|z - z| > r - \rho$ for all $z$ on $|z - z_0| = r$

$\therefore |R_n| \leq \frac{\rho^n}{2\pi} \frac{2\pi r M}{(r - \rho) r^n} = \frac{r M (\rho^n)}{r - \rho}

But $\frac{\rho}{r} < 1 \therefore R_n \to 0$ as $n \to \infty$

and $f(z)$ is represented by a Taylor series for all such $z$. 
When $z_0 = 0$, the series representation is called a Maclaurin series. 

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (z-0)^n$$

Example. Find Maclaurin series representation of

1) $f(z) = e^z$  
   $\Rightarrow f^{(n)}(z) = e^z$  
   so $f^{(n)}(0) = 1$  

   and  
   $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$  
   entire function valid in all $\mathbb{C}$

2) $f(z) = z^2 e^{3z}$

   $$z^2 e^{3z} = z^2 \sum_{n=0}^{\infty} \frac{(3z)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n z^{n+2}}{n!}$$

   let $n' = n+2$  
   $n' = 2$ when $n=0$

   $$z^2 e^{3z} = \sum_{n' = 2}^{\infty} \frac{3^{n'-2}}{(n'-2)!} z^{n'}$$  
   entire function valid in $\mathbb{C}$. 


Up to now considered fons. analytic in a domain $R$.

**Example** Expand the function in

$$f(z) = \frac{1+2z^2}{z^3+z^5}$$

in a series involving powers of $z$.

$$\frac{1}{1+z^2} \left[ \frac{2(1+z^3)-1}{1+z^2} \right] = \frac{1}{z^3} \left( 2 - \frac{1}{1+z^2} \right)$$

This is not analytic at $z=0$ so cannot expand in a MacLaurin series about $z=0$.

But

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + z^8 + \cdots$$

$$1 \geq 1$$

... when $0 < |z| < 1$

$$f(z) = \frac{1}{z^3} \left( 2 - 1 + z^2 - z^4 + z^6 - z^8 + \cdots \right)$$

$$= \frac{1}{z^3} + \frac{1}{z} - \frac{z + z^3 - z^5 + \cdots}{z^3}$$

looks like regular finite no. of terms MacLaurin Expansion in negative powers of $z$

... leads to Laurent Series expansion.
Laurent Series

Laurent series failed to be analytic at $z = 0$
so Taylor’s Theorem cannot be applied here.

Can often find a series representation of $f(z)$ involving
both negative and positive powers of $(z - z_0) \to$ Laurent series

**Theorem:** Suppose that $f(z)$ is analytic throughout
an annular region $R_1 < |z - z_0| < R_2$ and let $C$
denote any positively oriented simple closed contour
around $z_0$ and lying in that domain.

Then at each point $z$ in
the domain, $f(z)$ has the series representation

\[
    f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}
\]

\[
    = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n
\]

$R_1 < |z - z_0| < R_2$
where
\[ a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz, \quad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz \]

\[ n=0, 1, 2, \ldots \]
\[ n=1, 2, \ldots \]

or
\[ c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz \]

\[ (n=0, \pm 1, \pm 2, \ldots) \]

This series is called a Laurent series.

Notice that the principal part (negative powers of \((z-z_0)\)) isolates the singularities or poles of \(f(z)\), i.e., if no poles present we recover Taylor series expansion!

Laurent series e.takes care of isolated singularities of \(f(z)\)!
Proof: Apply Cauchy's theorem for multiply connected domains.

\[
\int_{C_2} \frac{f(z)}{z-z_0} \, dz - \int_{C_1} \frac{f(z)}{z-z_0} \, dz - \int_{\gamma} \frac{f(z)}{z-z_0} \, dz = 0
\]

Hence,

\[
2\pi i f(z_0) = \int_{C_2} \frac{f(z)}{z-z_0} \, dz - \int_{C_1} \frac{f(z)}{z-z_0} \, dz
\]

\[
(3-z_0) \left[ 1 - \frac{z-z_0}{3-z_0} \right] < 1
\]

\[
\left| \frac{z-z_0}{3-z_0} \right| < 1
\]

On \( C_2 \), usual Taylor series.

First integral is identical to Taylor series expansion.
PS. 20.

Second Integral

\[
- \frac{1}{2\pi i} \int_{C_1} \frac{f(3)}{z-z_0} \, d3 = \frac{1}{z-z_0} \cdot \frac{1}{2\pi i} \int_{C_1} \frac{f(3)}{z(z-z_0)} \left[ 1 + \left( \frac{z-z_0}{z-z_0} \right) + \left( \frac{z-z_0}{z-z_0} \right)^2 + \cdots \right] \, d3
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{(z-z_0)^{(n+1)}} \cdot \frac{1}{2\pi i} \int_{C_1} \frac{f(3)}{(z-z_0)^{n+1}} \, d3
\]

But \( n' = n+1 \) or \( n = n' - 1 \) \( \rightarrow n = -n' + 1 \)

\[
= \sum_{n'=1}^{\infty} \frac{1}{(z-z_0)^{n'}} \cdot \frac{1}{2\pi i} \int_{C_1} \frac{f(3)}{(z-z_0)^{-n'+1}} \, d3
\]

\[
= \sum_{n'=1}^{\infty} \frac{b_{n'}}{(z-z_0)^{n'}}
\]

\[
\therefore \quad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \left( \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \right)
\]

because \( C_1 \) can be replaced by arbitrary curve \( C \).

(by Cauchy's Theorem).
Exercise: Show that the remainder terms in each series go to zero or not so?
Example 1

Replace $z$ by $\frac{1}{z}$ in Maclaurin expansion

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots$$

to get

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

- contains no positive powers of $z$!

Note that coefficient $[C_{-1} = 1]$ (coef at $1/z$)

But $C_{-1} = \frac{1}{2\pi i} \oint_C e^{1/z} \, dz = 1$

Recall $\oint_C (z-z_0)^{-1} \, dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1, \text{ -residue} \end{cases}$