1. The differential equation
\[ x \frac{d^2 y}{dx^2} + (1 - x) \frac{dy}{dx} + ny = 0 \]
is known as Laguerre’s equation.

a) Obtain the regular solution in the form

\[ y_1(x) = B_0 \left[ 1 + \sum_{k=1}^{\infty} \frac{(-1)^k n(n-1)(n-2) \cdots (n-k+1)}{(k!)^2} x^k \right] \]

b) Show that this solution is a polynomial of degree \( n \) when \( n \) is a nonnegative integer, and verify that the choice \( B_0 = 1 \) leads to the Laguerre polynomial of degree \( n \), with the definition

\[ L_n(x) = 1 - \binom{n}{1} \frac{x}{1!} + \binom{n}{2} \frac{x^2}{2!} - \cdots + \frac{(-x)^n}{n!}, \]

Where \( \binom{n}{k} \) represents the binomial coefficient \( n!/[n!(n-k)!] \).

2. The differential equation
\[ \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2ny = 0 \]
is known as Hermite’s equation.

Obtain the general solution in the form \( y(x) = c_1 u_n(x) + c_2 v_n(x) \), where

\[ u_n(x) = 1 - \frac{n}{1!} \frac{x^2}{1!} + \frac{n(n-2)}{1 \cdot 3} \frac{x^4}{2!} - \frac{n(n-2)(n-4)}{1 \cdot 3 \cdot 5} \frac{x^6}{3!} + \cdots \]

and

\[ v_n(x) = x - \frac{n-1}{3} \frac{x^3}{1!} + \frac{(n-1)(n-3)}{3 \cdot 5} \frac{x^5}{2!} - \frac{(n-1)(n-3)(n-5)}{3 \cdot 5 \cdot 7} \frac{x^7}{3!} + \cdots . \]

[Hence verify that the solution \( u_n(x) \) is a polynomial of degree \( n \) when \( n \) is a positive even integer or zero, whereas \( v_n(x) \) is a polynomial of degree \( n \) when \( n \) is a positive odd integer. That multiple of the \( n \)-th-degree polynomial for which the coefficient of \( x^n \) is \( 2^n \) is called the \( n \)-th Hermite polynomial and is often denoted by \( H_n(x) \).]
3. a) Show that the equation

\[ x \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0 \]

possesses equal exponents \( s_1 = s_2 = 0 \) at \( x = 0 \)

b) Obtain the regular solution, and denote by \( u_1(x) \) the result of setting the leading coefficient \( A_0 \) equal to unity.

c) Assume a second solution of the form

\[ y_2(x) = C u_1(x) \log x + v(x), \]

Where \( c \neq 0 \), and show that \( v(x) \) must satisfy the equation

\[ x \frac{d^2v}{dx^2} + \frac{dv}{dx} - v = -2C \frac{d}{dx} u_1 \]

d) Obtain one solution of this equation in the form

\[ v(x) = \sum_{k=0}^{\infty} B_k x^{k+s_2} = \sum_{k=0}^{\infty} B_k x^k, \]

showing that \( C \) and \( B_0 \) are arbitrary, but taking \( C = 1 \) and \( B_0 = 0 \) for convenience. Hence obtain the general solution of the original equation in the form

\[ y = c_1 u_1(x) + c_2 [u_1(x) \log x + v(x)]. \]
522 Extra HW question:

4. Use a Frobenius series to solve the second order differential equation

\[ x(1-x) \frac{d^2 y}{dx^2} + [\gamma - (\alpha + \beta + 1)x] \frac{dy}{dx} - \alpha \beta y = 0 \]

Where \( \alpha, \beta, \gamma \) are parameters. Determine the indicial exponents and show that both are equal if the parameter \( \gamma = 1 \).

Obtain the recurrence relation for the Frobenius series coefficients and show that a solution can be expressed in the form:

\[ F(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha \beta}{1 \gamma} x + \frac{\alpha (\alpha + 1) \beta (\beta + 1)}{1 \times 2 \gamma (\gamma + 1)} x^2 + \cdots \]

When the indicial exponent is zero, \( F \) is called the hypergeometric function. Show that this expression reduces to the standard geometric series when \( \alpha = 1, \beta = \gamma \).