Math 422/522 – Spring 2018 - HOMEWORK 13

Question 1:

Suppose that a rod is such that heat escapes from the lateral boundary according to Newton’s law of cooling, so that $T(x, t)$ satisfies the equation

$$
\alpha^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} + \beta(T - T_0),
$$

where $\beta$ is a constant and $T_0$ is the temperature of the surrounding medium.

The initial temperature distribution is $T(x, 0) = f(x)$ and the ends $x = 0$ and $x = l$ are maintained at $T_1$ and $T_2$, respectively, when $t > 0$.

(a) Show that the substitution

$$
T(x, t) = T_0 + U(x, t)e^{-\beta t}
$$

reduces the problem to the following one:

$$
\frac{\partial^2 U}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial U}{\partial t},
$$

$$
U(x, 0) = f(x) - T_0, \quad U(0, t) = (T_1 - T_0)e^{\beta t}, \quad U(l, t) = (T_2 - T_0)e^{\beta t}.
$$

(b) In the important special case when $T_1 = T_2 = T_0$, obtain the solution of the original problem in the form

$$
T(x, t) = T_0 + \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} e^{-\frac{\beta + (n^2 \pi^2 \alpha)}{\alpha^2} t},
$$

where

$$
a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx - \frac{1 - \cos \frac{n\pi}{\alpha^2} T_0}{\frac{\alpha^2}{n\pi}} 2T_0.
$$

Question 2:

A rod of length $l$, with insulated lateral boundaries, has the end $x = 0$ maintained at $T = T_1$ when $t > 0$, whereas heat escapes through the end $x = l$ according to Newton’s law of cooling in the form

$$
\left[ h \frac{\partial T}{\partial x} + (T - T_0) \right]_{x=l} = 0,
$$

where $T_0$ is the temperature of the surrounding medium and $h$ is a constant. The initial temperature distribution along the rod is prescribed as $T(x, 0) = f(x)$.

(a) Obtain the steady-state distribution in the form

$$
T_s = T_1 - \frac{T_1 - T_0}{1 + h/\alpha} x.
$$

(b) Show that the transient distribution can be assumed in the form

$$
T_T(x, t) = \sum_{n=1}^{\infty} a_n \sin \frac{k_n x}{l} e^{-k_n \alpha t / \alpha^2},
$$

where $k_n$ is the $n$th positive root of the equation $\tan k + hk = 0$.

(c) Obtain the required temperature distribution in the form $T = T_s + T_T$, where $a_n$ is determined by the relation

$$
a_n \int_0^l \frac{k_n^2 x}{l^2} dx = \int_0^l [f(x) - T_s(x)] \sin \frac{k_n x}{l} dx.
$$