1. (a) If $\alpha$ is a positive real constant, determine the Fourier sine and cosine integral representations of $e^{-\alpha x}$ in the forms

$$e^{-\alpha x} = \int_0^\infty \frac{u}{\pi} \left( \frac{\alpha}{\alpha^2 + u^2} \right) \sin u x \, du = \int_0^\infty \frac{\alpha}{\pi} \left( \frac{\alpha}{\alpha^2 + u^2} \right) \cos u x \, du \quad (\alpha > 0, \ x > 0).$$

(b) Use these results to determine functions $A(u)$ and $B(u)$ such that

$$\frac{x}{\alpha^2 + x^2} = \int_0^\infty A(u) \sin u x \, du, \quad \frac{\alpha}{\alpha^2 + x^2} = \int_0^\infty B(u) \cos u x \, du \quad (\alpha > 0, \ x > 0).$$

(c) Deduce from the preceding results that

(*Note: $S =$ Sine Fourier Transform, below, and $C =$ Cosine Fourier Transform)

$$S\left\{ \frac{x}{\alpha^2 + x^2} \right\} = \frac{\pi}{2} e^{-\alpha x}, \quad S\{e^{-\alpha x}\} = \frac{u}{\alpha^2 + u^2} \quad \text{and}$$

$$C\left\{ \frac{\alpha}{\alpha^2 + x^2} \right\} = \frac{\pi}{2} e^{-\alpha x}, \quad C\{e^{-\alpha x}\} = \frac{\alpha}{\alpha^2 + u^2} \quad \text{when } \alpha > 0.$$

2. Show that $\alpha$ can be replaced by $a + ib$, where $a$ and $b$ are real and $\alpha > 0$, in Problem 1(a). In particular, show that $e^{-\alpha \left( \cos bx - i \sin bx \right)} = \frac{2}{\pi} \int_0^\infty \frac{u \sin u x}{(\alpha^2 - b^2 + u^2) + 2iab} \, du$

when $\alpha > 0$ and $x > 0$, by equating real and imaginary parts of the equal members, deduce the sine integral representations of $e^{-\alpha \cos bx}$ and $e^{-\alpha \sin bx}$ when $\alpha > 0$.

3. (a) If $\alpha$ is a positive real constant, determine the complex form of the Fourier integral representation of the $e^{-\alpha x} = \int_0^\infty \left( \frac{\alpha}{\pi} \frac{1}{\alpha^2 + u^2} \right) e^{iu x} \, du \quad (\alpha > 0).$

(b) Deduce from this result the function $C(u)$ such that

$$\frac{\alpha}{\alpha^2 + x^2} = \int_0^\infty C(u) e^{iu x} \, du \quad (\alpha > 0).$$

(c) Deduce that

(*Note: $F =$ Fourier Integral Transform, below)

$$F\left\{ \frac{\alpha}{\alpha^2 + x^2} \right\} = \pi e^{-\alpha |u|}, \quad F\{e^{-\alpha x}\} = \frac{2\alpha}{\alpha^2 + u^2} \quad \text{when } \alpha > 0.$$