Multiply - Connected Regions

Consider region where \( f(z) \) is analytic in a domain bounded by \( C_0 \) and \( C_1 \). 
\( Z_0 \) may or may not be a singularity.

Cauchy-Goursat Theorem

\[
\oint_{C_0} f(z)\,dz = \oint_{C_1} f(z)\,dz
\]

Proof:

Make a cut from \( C_0 \) to \( C_1 \).

Now region enclosed by \( C_1, C_0, C_2, C_3 \) encloses a region in which \( f(z) \) is analytic.

Apply Cauchy-Goursat Th to this region:

\[
\oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz + \oint_{C_0} f(z)\,dz = 0
\]

\( C_1 \parallel C_3 \parallel C_0 \parallel C_2 \)

\( -\oint_{C_1} f(z)\,dz = \text{reverse direction of arrows on } C_1 \)

Contributions on \( C_2 \) and \( C_3 \) are equal but opposite in sign \( \rightarrow \) cancel.

\[
\oint_{C_1} f(z)\,dz = \oint_{C_0} f(z)\,dz
\]

This result is important as one can continuously shrink \( C_0 \) down onto \( C_1 \) without changing the result.

This is often referred to

This idea can be extended to multiple regions contained within \( C_0 \).

- Use \( Z_0 \) for \( C_0 \) on next pages.
Extension of Cauchy's Theorem to multiply connected domains.

**Theorem:** Let the inside of the pws Jordan curve $\gamma_0$ contain the disjoint piecewise smooth Jordan curves $\gamma_1, \gamma_2, \ldots, \gamma_n$, none of which is contained inside another.

Suppose $f(z)$ is analytic in a region $\Omega$ containing the set $S$ consisting of all points on and inside $\gamma_0$ but not inside $\gamma_k$, $k=1, 2, \ldots, n$. Then

$$\int_{\gamma_0} f(z)\,dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz.$$
Proof

Connect \( x_k \) to \( x_{k+1} \) with paths and arcs \( L_k \), \( k=0,1,\ldots,n-1 \) and \( x_n \) to \( x_0 \) with \( L_n \).

Note the reversal of path on \( x_k \), \( k=1,2,\ldots,n \).

Now the upper and lower regions are decomposed into 2 simply connected domains where \( f(z) \) is analytic \( \Rightarrow \) Cauchy's theorem is satisfied in each domain. (as each curve is traversed in positive sense).

However, the integrals along \( L_k \), \( k=0,1,\ldots,n \) are evaluated in opposite sense so these cancel out \( \int_{L_k} f(z) \, dz = -\int_{L_k} f(z) \, dz \).
and the integrals over each $x_k$, $k = 1, 2, \ldots, n$ are the negation of the positively oriented integrals.

... adding together the individual pieces

$$\int_{x_0} f(x) \, dx - \sum_{k=1}^{n} \int_{x_k} f(x) \, dx = 0$$

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Note we have used in the proof

$$\int_{-c} f(x) \, dx = -\int_{c} f(x) \, dx$$

and

$$\int_{x_1+x_2} f(x) \, dx = \int_{x_1} f(x) \, dx + \int_{x_2} f(x) \, dx$$

when $x_1, x_2$ are piecewise continuous Jordan arcs.

This result is crucial...